Random spreading phenomena in annealed small world networks

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We study the simple random walk dynamics on an annealed version of a Small-World Network (SWN) consisting of N nodes. This is done by calculating the mean number of distinct sites visited $S(n)$ and the return probability $P_{00}(t)$ as a function of the time t. $S(t)$ is a key quantity both from the statistical physics point of view and especially for characterizing the efficiency of the network connectedness. Our results for this quantity shows features similar to the SWN with quenched disorder, but with a crossover time that goes inversely proportianal to the probability p of making a long range jump instead of being proportional to p^{-2} as in quenched case. We have also carried out simulations on a modified annealed model where the crossover time goes as p^{-2} due to specific time dependent transition probabilities and we present an approximate self-consistent solution to it.
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I. INTRODUCTION

During the last few years an overwhelming amount of evidence has been accumulated about diverse networks showing “small world” properties. This means that on the average an arbitrarily selected node can be reached from another node in very few steps despite the fact that only relatively small number of links are present [1]. Such networks include social nets [2], the internet [4], the www of linked documents [3], scientific cooperation relations [5] etc. Watts and Strogatz were the first to suggested a simple mathematical model, which reflects the small world phenomenon: Take a regular lattice and introduce (few) random links between any pairs of sites [6]. This Small World Network (SWN) model interpolates between regular graphs or lattices and the so called Erdős-Rényi random graphs [1].

Real networks are commonly characterized by a number of parameters but perhaps the most important measure is the average distance between their nodes. It has turned out that there is a rich family of models with the small world property but they differ in many other respects. For example, the degree distribution of the nodes is Poissonian for the SWN while many real networks are often scale free (with a power law decay in the degree distribution) for which preferential growth models have been introduced [7]. Thus SWN's are very interesting graphs not only because of these properties but also because they are simple with nontrivial behavior but provide possibilities for exact solutions [9] and they seem to have a direct application in polymer physics [10].

I addition to static structural properties of networks there is ever growing interest in the dynamics processes in networks. In spite of this, relatively few results have sofar been published on the dynamics of small world models. It is expected that the underlying network topology should have a major impact on practically any phenomenon taking place on it. This is supported by the recent results on the the spectral density of the adjacency matrix of small world models, which show that these graphs produce a dramatic deviation from the semi-circle law of random graphs [11].

It is our belief that spreading phenomena are perhaps the most direct examples of dynamical processes reflecting the small world properties. In direct spreading of e.g. a disease the nodes of the graph get infected by the rule that infection propagates each time step to all uninfected neighbors of already infected sites [12]. Then the simplest example of non-trivial dynamics could be that of a diffusing particle on the SWN. This in turn is related to the intensively studied process of random walks in random environments [13]. Recently some related papers have been published, for example the study of spectral properties of the Laplacian on the SWN [18]. While some numerical and analytic results were given in [16] for the spreading phenomenon being characterized by the average access time to the sites of the system, Jasch and Blumen [14] published simulation results for spreading on SWN using random walk dynamics with the main quantity of interest being the average number of distinct sites vistited at a given time.

In all these studies a simple version of the SWN was considered: A one-dimensional ring consisting of N nodes with k-neighbor interactions was assumed where, in addition, Np new links were introduced between arbitrarily chosen, but not yet connected nodes. The result by Jasch and Blumen [14] is intuitively very clear: whenever there is however small ratio of long range links present in the graph, $S(t)$ changes its asymptotic behavior from the $\sim \sqrt{t}$ relationship characteristic for the one-dimensional case to $\sim t$ representing the long range interconnected graph. In [14] a crossover formula was presented by relating the probability of long range links p to a character-

istic time t^*, where the crossover sets in ($p^\alpha t^*$ =const). Again, one would intuitively expect that $\alpha = 2$ and that the numerically found discrepancies were occuring most probably due to limited system sizes. In fact we found [15] in our numerical simulations that $\alpha = 2$. This turned out to be true not only for $S(n)$ but also for the so called return probability P_{00} [15,17].

In their original form the SWN is defined with quenched disorder represented by the long range links. However, it is interesting to ask to what extent the small world properties of the dynamics depend on the quenched character of disorder. In order to find some answers we are in this paper setting to study the annealed version of the dynamics on SWN's, where the links are not frozen in but the walker has at any site the choice to make either a long range jump (with probability p) or to continue the random walk in the regular part of the lattice (with probability $(1-p)$). Such a model was studied in this context by Pandit and Amritkar [16], where the average access time was calculated. Here we will concentrate to determine through analytical theory and simulations the average number of distinct sites visited and the return probability.

The paper is organized as follows: In the next section we introduce the model and present the solution. In Section 3 we discuss a modified version of the model where the crossover time scales as p^{-2} as in the quenched case. Finally we present a discussion and a summary.

II. THE BASIC MODEL AND ITS SOLUTION

In order to visualize the model let us consider a ring with first and second neighbor connections ($k = 2$), the sites of which being numbered from 0 to $N-1$. In this system we let a random walker to start at site 0 at time $t = 0$. Before the first step sets in the four links of site 0 are rewired by moving the other endpoint to any of the other sites with probability p. Then the walker moves to one of the sites newly linked to its position with equal probability. Let us call this site i. After the walker arrives the site i the original lattice is restored, the four links for i are rewired as before, etc. In this way the small world property is maintained but we have got rid of the problems of quenched randomness.

The movement of the random walk is governed by the simple master equation:

$$\partial_t P_i(t) = \sum_{j=1,N} T_{ij} P_i(t) \tag{1}$$

where the continuum time limit has been applied. Here $P_i(t)$ is the probability that the walker is at site i at time t and

$$T_{ij} = W_{ij} - \delta_{ij} \tag{2}$$

with W_{ij} being the transition matrix, which has the following form:

$$\mathbf{W} = (1-p)\mathbf{W}^{(S)} + p\mathbf{W}^{(L)}. \tag{3}$$

Here (S) and (L) refer to short and long range jumps, respectively. The zeroth row of $\mathbf{W}^{(S)}$ reads as follows

$$W^{(S)}_{0\,\cdot} = \frac{1}{2k}(0, \underbrace{1,\ldots,1}_{k \text{ times}}, \underbrace{0,\ldots,0}_{N-2k-1 \text{ times}}, \underbrace{1,\ldots,1}_{k \text{ times}}). \tag{4}$$

A similar equation holds for the $\mathbf{W}^{(L)}$:

$$W^{(L)}_{0\,\cdot} = \frac{1}{N-2k-1}(\underbrace{0,\ldots,0}_{k+1 \text{ times}}, \underbrace{1,\ldots,1}_{N-2k-1 \text{ times}}, \underbrace{0,\ldots,0}_{k \text{ times}}). \tag{5}$$

The i^{th} rows are then obtained by cyclically shifting the 0^{th} rows to the right. All $\mathbf{W}$ and $\mathbf{T}$ matrices have the Toeplitz form, i.e., T_{ij} depends only on the difference $(i-j)$. Therefore, the right hand side of Eq. (1) is a convolution which leads after spatial Fourier transform to the following form

$$\partial_t \hat{P}_q(t) = (\hat{W}_q(t) - 1)\hat{P}_q(t). \tag{6}$$

With the initial condition

$$P_i(0) = \delta_{0i} \tag{7}$$

the formal solution is as follows

$$\hat{P}_q(t) = \exp[\int_0^t (\hat{W}_q(u) - 1)du]. \tag{8}$$

This solution can be easily evaluated for the matrix $\mathbf{W}$ given in Eqs (3)-(5).

A. Return probability and spreading rate

Let $F_{ij}(t)$ denote the probability of the random walker visiting site j at time t having started from site i. Then we can write

$$P_{ij}(t) = \int_0^t F_{ij}(u) P_{jj}(t-u) du, \tag{9}$$

where $P_{ij}(t)$ is the probability for the random walker to move from i to j during the time t. From this we get through the Laplace transform the following equation

$$\tilde{F}_{ij}(z) = \frac{\tilde{P}_{ij}(z)}{\tilde{P}_{jj}(z)}. \tag{10}$$

Now let us take $s(t)$ as the probability of observing a new site, or as the *spreading rate* at time t when the random walker started from site 0:

$$s(t) = \sum_{i=0}^{N-1} F_{0i}(t) \tag{11}$$

By taking the *return probabilities* $P_{ii}(t)$ to be the same for all i the equation (11) reads as follows

$$\tilde{s}(z) = \frac{1}{\tilde{P}_{00}(z)} \sum_{i=0}^{N-1} \tilde{P}_{0i}(z) = \frac{1}{z\tilde{P}_{00}(z)} \tag{12}$$

As stated before the quantity of interest is the average number of distinct sites visited, $S(t)$, which is expressed as

$$S(t) = \int_0^t s(u)du. \tag{13}$$

Then by using Eq. (12) we obtain $S(t)$ by the inverse Laplace transform of the function $\tilde{s}(z)/z$. It is noted that $S(t)$ could physically be intepreted as the amount of spreading the random walker has covered at time t.

B. Results

In order to obtain the return probability and the mean number of visited sites as a function of the probability p we have solved numerically the set of equations (8) and (12) given above. For the purpose of checking the validity of these solutions we compare them with discrete time simulations both for $P_{00}(t)$ and $S(t)$. As depicted in Figs 1 and 3 the agreement for both these quantities is excellent, apart from the short time behavior, which is a consequence of the difference between the discrete and continuum time versions of the system.

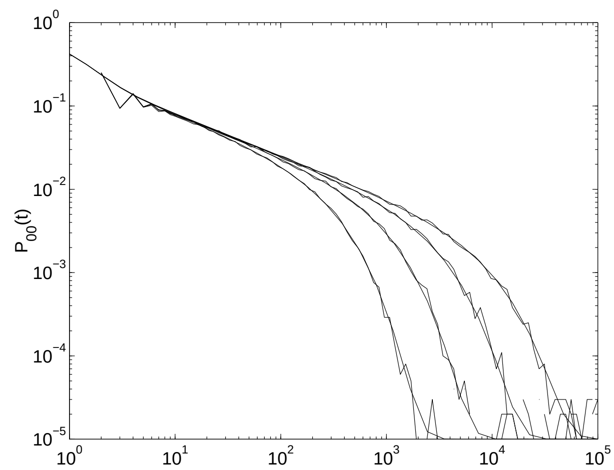

FIG. 1. The simulation results for the return probabilities for $p = 10^{-4}$, $10^{-3.5}$, 10^{-3}, $10^{-2.5}$ (uneven line). The smooth curve shows the result of the analytical theory.

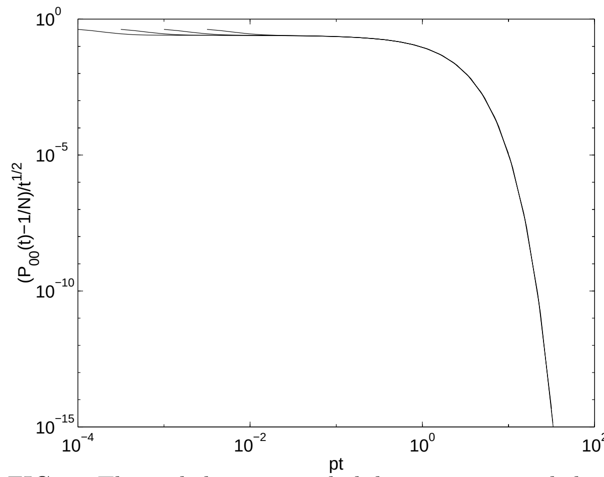

FIG. 2. The scaled return probabilities against scaled time for $p = 10^{-4}$, $10^{-3.5}$, 10^{-3}, $10^{-2.5}$.

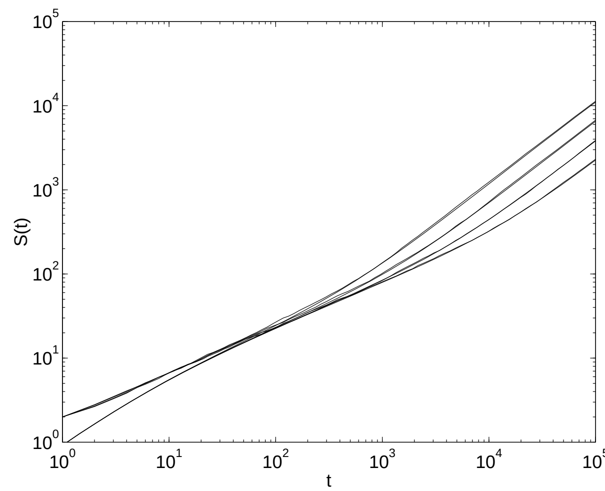

FIG. 3. The simulation results for the mean number of distinct sites visited $S(t)$ (i.e. the amount of spreading) for $p = 10^{-4}$, $10^{-3.5}$, 10^{-3}, $10^{-2.5}$. These curves start from $S(0) = 1$. Analytical results, which start at $S(0) = 0$, are also shown.

Fig. 1 shows the return probability P_{00} as a function of time. It is clearly evident from this figure that the simulation results agree very well with analytical calculations. In Fig. 2 we present a scaling plot of this data, i.e. $(P_{00} - 1/N)\sqrt{t}$ vs. scaled time pt, where $1/N$ represents the long time limit. This scaling obeys the following functional form:

$$P_{00}(t) = t^{-1/2}\varphi(tp^{\alpha}) \tag{14}$$

where φ is a universal scaling function with the the property that $\varphi(x) \propto const$ for $x \ll 1$ and it decays rapidly for $x > 1$. The exponent α is unity (see Fig. 2) which is easily understood, because the short time characteristic for the one-dimensional case is valid until the walker makes a long jump. This takes the time $1/p$, on the average, which goes as a characteristic time into the argument of the function f. This is in contrast with the

behavior of the quenched model, where in order to make a long jump, the distance proportional to $1/p$ has to be abbridged by diffusive motion on a one-dimensional topology of the system leading to a characteristic time behavior of $\sim (1/p)^2$ [14,15].

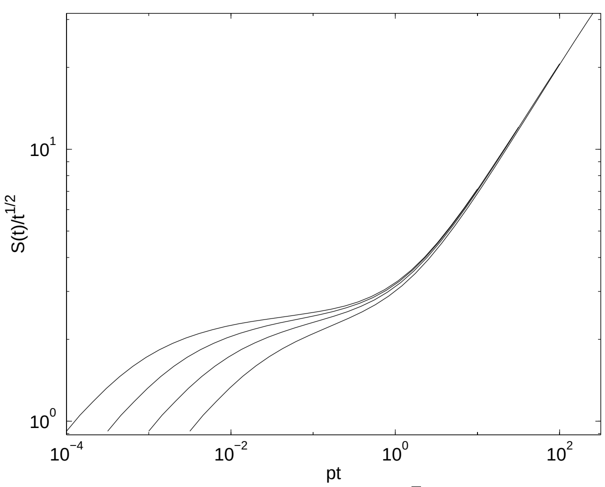

FIG. 4. The scaled spreading $(S(t)/\sqrt{t})$ against the scaled time (pt) for $p = 10^{-4}$, $10^{-3.5}$, 10^{-3}, $10^{-2.5}$.

In Fig. 3 we show the simulation and analytical results for the mean number of distinct sites visited, i.e. $S(t)$ as a function of time. Again, the good agreement between the simulation results and theoretical analysis is clear. The scaling form corresponding to (14) would now read as

$$S(t) = t^{1/2} f(tp^{\alpha}) \tag{15}$$

where the universal scaling function f is of the following form

$$f(x) \propto \begin{cases} \text{const} & \text{for} \quad x \ll 1 \\ \sqrt{x} & \text{for} \quad x \gg 1 \end{cases} \tag{16}$$

Here the short time behavior is $S \sim t^{1/2}$ while for long times it becomes proportional to t. As seen from the scaling plot of Fig. 4, the same $\alpha = 1$ applies to this case as for the return probability.

III. THE SELF–CONSISTENT MODEL

Next we ask the question whether it is possible to modify the annealed model in a way that the crossover is shifted such that $\alpha = 2$ would be obtained. This behavior is achieved by assuming a time dependent transition probability, as discussed below.

Since the scaling of the transition occurs in the quenched system later (as $\sim p^2 t$) we replaced the multiplier p of $\mathbf{W}^{(L)}$ in Eq. (3) with $ps(t)$ to simulate the situation where the walker has a probability of making a long range leap only when visiting a previously unseen site:

$$\mathbf{W} = (1-p)\mathbf{W}^{(S)} + ps(t)\mathbf{W}^{(L)}. \tag{17}$$

The corresponding master equation cannot be solved explicitly but we can still estimate the solution with arbitrary accuracy by numerical iteration scheme. In Fig. 5 we show the resulting time dependent behavior of the random walk spreading for the self-consistent model and simulated quenched system. Apart from the short times the agreement between these results seem to be quite good.

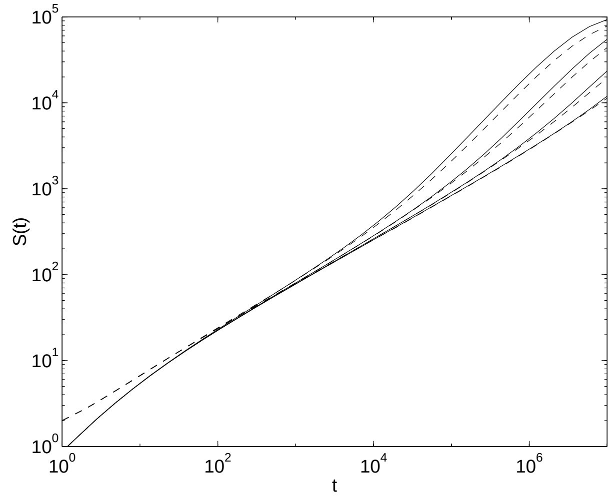

FIG. 5. The results for the spreading vs. time of the self–consistent model obtained from the analytical theory (solid line) and from the quenched simulations (dashed line) for $p = 10^{-4}$, $10^{-3.5}$, 10^{-3}, $10^{-2.5}$.

In Fig. 6 we show the scaling plot with $\alpha = 2$ for the self-consistent model. Apart from early times the scaling seems to hold.

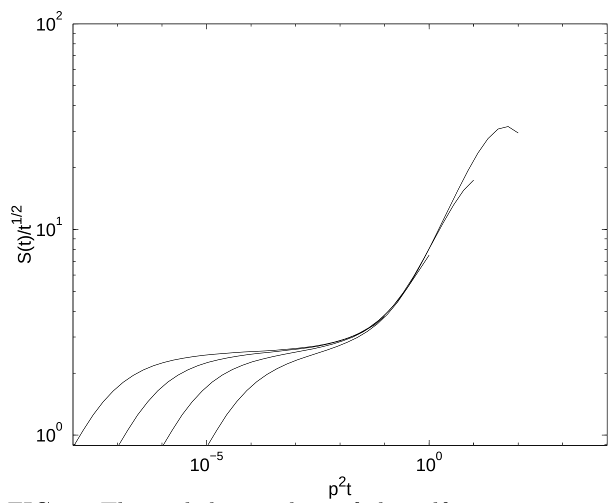

FIG. 6. The scaled spreading of the self-consistent model $S(t)/\sqrt{t}$ against scaled time $p^2 t$ for $p = 10^{-4}$, $10^{-3.5}$, 10^{-3}, $10^{-2.5}$.

Hence the numerical results justify the choice of Eq. (17) which reflects the fact that in the quenched model one

dimensional random walk has to be carried out between two long jumps.

IV. SUMMARY

We have shown that the annealed random walk model with rare long jumps reflects some aspects of the quenched SWN's. In the simplest case, with time independent transition probabilities, the model can be solved. However, as expected, only qualitative agreement between the quenched and the annealed models can be observed. With properly chosen time dependent transition probabilities we obtain even the proper exponent $\alpha = 2$. We think that it is quite interesting that the behavior of the spreading can be estimated by an annealed model.

ACKNOWLEDGEMENTS

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